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PARTICIPATION AND THE PROVISION OF DISCRETE PUBLIC GOODS: A STRATEGIC ANALYSIS

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ABSTRACT

This paper considers the Nash equilibria to a game where a discrete public good is to be provided. Each individual may participate by making a fixed contribution. If a sufficient number of contributions are made, the good is provided. Otherwise, the good is not provided. One variant of the rules allows for contributions to be refunded when the good is not provided. For pure strategies, we find that the Nash equilibria with a refund are a superset of those without a refund. For both rules, the efficient number of players contributing is an equilibrium. For mixed strategies, to every equilibrium without a refund, there is a corresponding equilibrium with a refund with a higher number of expected contributors. Mixed strategy equilibria "disappear" as the number of players grows large. Some results reported in the experimental literature are discussed in light of these theoretical results.

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I. INTRODUCTION*

Participation games are games in which members of a team pursue a common but costly goal. The decision problem the group faces is how to share the costs of achieving the goal. The group must confront the well-known incentive problem where each member of the group is tempted to free ride, hoping that other members of the group will pay the cost. In Palfrey and Rosenthal (1982) we formally defined participation games and analyzed games with two teams that are akin to elections and referenda.

In these games each member of each team had to decide whether to vote. If one team had more members voting than did the other team, its members each received a reward of one. The members of the losing team each got zero. However, it cost c, $0 < c < 1$ to vote. The basic payoffs were

1	Non-voters on winning team
1-c	Voters on winning team
0	Non-voters on losing team
-c	Voters on losing team.

Two alternative rules for dealing with ties were explored. For both rules, we obtained identical asymptotic results. With large number of voters, we found only two types of Nash equilibria; one where nearly no one voted and another in which turnout was approximately twice the membership of the smaller team.

An important special case of these team games arises when there is only one team. The game then models a discrete public good problem. If at least one member of the team contributes (votes) the good (worth 1 to everyone) is provided. Otherwise, everyone receives zero.[1]

More general versions of one team games with discrete public goods and fixed contributions occur in the real world. Some examples follow. (A) A new coffee pot for an office of 50 employees costs $30. If at least 30 employees contribute $1, the purchase will be made. (In this case, transaction costs might preclude requests for variable contributions.) (B) If 10 percent of the registered voters sign a petition at the city hall, a referendum is held on some issue. (C) Fifty people see an assault taking place. If one person calls the police, the victim is saved.

In this paper, we formally generalize the one team game as follows.

1. There are M players i each with pure strategies, denoted s_i, of either contributing a fixed amount ($s_i = 1$) or not contributing at all ($s_i = 0$).

We denote the number of contributors necessary to produce the public good by w and the actual number of contributors by m. We consider two slightly different "rules" of the game, <u>no refund</u> ($\bar{R}$) and refund (R).

2. ($\bar{R}$) If $m \geq w$ and $s_i = 1$, i receives 1-c
If $m \geq w$ and $s_i = 0$, i receives 1
If $m < w$ and $s_i = 1$, i receives -c
If $m < w$ and $s_i = 0$, i receives 0

Under the no refund rule, a player who contributes does not get a refund if the public good isn't provided. This is contrasted with a simple refund rule in which contributions are fully refunded if the public good is not provided.

2'. (R) If $m \geq w$ and $s_i = 1$, i receives 1-c
$m \geq w$ and $s_i = 0$, i receives 1
If $m < w$ and $s_i = 1$ or 0, i receives 0.

Version (2'.) was suggested by Simmons et al. (1981) and van de Kragt et al. (1982), following earlier work by Brubaker (1975), Coombs (1973), and Bohm (1971). An interpretation of their game is that the players are asked to contribute toward the public good.[2] They are told that if at least w contributions are made, the good will be produced. Otherwise, all contributions will be refunded. These authors interpret the removal of the -c payoff from (2.) as removing the "fear" motivation for free riding. Whereas (2.) has both a "greed" and a "fear" motivation, (2'.) has only a "greed" motivation. Their conjecture is that the refund rule will produce superior allocations to the no refund rule because the free rider problem is

partially overcome in this way. In an attempt to formalize and verify this conjecture, we eschew these psychological connotations and simply analyze and compare the Nash equilibria of both games. In addition to characterizing the equilibria, we examine limiting properties with a large number of players. We then use our results to analyze the data from two sets of experiments.

II. EQUILIBRIUM ANALYSIS

We analyze equilibria that can be characterized by a partition of the M players into three groups:

G^1 If $i \in G^1$, $s_i = 1$, so i contributes
G^2 If $i \in G^2$, $s_i = 0$, so i does not contribute
G^3 If $i \in G^3$, i has $s_i = 1$ with probability q, so i uses a mixed strategy.

Note that we impose symmetry in mixed strategies. There is no i subscript in G^3.

Case I. No Refund ($\bar{R}$).

Ia. $|G^3| = 0$. (Pure strategy equilibria.)

Proposition 1. If $w = 1$, there are M pure strategy equilibria, each with one contributor and M-1 non-contributors. If $w > 1$, there are $\binom{M}{w}$ pure strategy equilibria each with w contributors and one equilibrium with no contributors. These equilibria exhaust the pure strategy equilibria.

<u>Proof</u>: Obvious.

<u>Ib</u>. $|G^3| \neq 0$. (Symmetric mixed strategy equilibria.) To analyze this case requires the following definitions:

$k = |G^1|$

$j = |G^2|$

m^3 = Number of contributors in G^3,

and, for $i \in G^3$,

m_i^3 = Number of contributors other than i in G^3.

The equilibrium condition requires that each player be indifferent between contributing and not contributing. The expected payoff if i does not contribute equals the value of the good (1) times the probability that contributions by others are sufficient to produce the good. That is, for a member i of G^3, the expected payoff of not contributing equals

$$\text{prob}\{m_i^3 \geq w-k\}.$$

Similarly, the expected payoff to i of contributing is

$$\text{prob}\{m_i^3 \geq w-k-1\} - c$$

The equilibrium condition for each member of G^3 (mixers) requires that the player be indifferent between contributing and not contributing. That is

$$\text{prob}\{m_i^3 \geq w-k\} = \text{prob}\{m_i^3 \geq w-k-1\} - c$$

The equilibrium condition for each member of G^1 (contributors) is that the payoff to contributing is at least as great as the payoff to not contributing. That is

$$\text{prob}\{m^3 \geq w-k+1\} \leq \text{prob}\{m^3 \geq w-k\} - c.$$

Similarly, for members of G^2 (noncontributors) we have

$$\text{prob}\{m^3 \geq w-k\} \geq \text{prob}\{m^3 \geq w-k-1\} - c.$$

The three conditions outlined above can be written in algebraic form, in terms of w, j, k and M. These are given below

$$(G^1) \qquad c \leq \text{Prob}(m^3 = w-k) = \binom{M-k-j}{w-k} q^{w-k}(1-q)^{M-w-j} \tag{1.1}$$

$$(G^2) \qquad c \geq \text{Prob}(m^3 = w-k-1) = \binom{M-k-j}{w-k-1} q^{w-k-1}(1-q)^{M-w-j+1} \tag{1.2}$$

$$(G^3) \qquad c = \text{Prob}(m_i^3 = w-k-1) = \binom{M-k-j-1}{w-k-1} q^{w-k-1}(1-q)^{M-w-j} \tag{1.3}$$

For some combinations of c,j,k,w,M, (1.1)-(1.3) cannot simultaneously be satisfied for any value of q between 0 and 1. Characterizing which types of equilibria exist is equivalent to characterizing when (1.1)-(1.3) can be simultaneously satisfied for some value(s) of q between 0 and 1. For completeness, we note three special cases in which (1.1)-(1.3) need not all be satisfied. If

$k = 0$, then only (1.2) and (1.3) need be satisfied; if $j = 0$ only (1.1) and (1.3) apply and if $k = j = 0$ only (1.3) applies.

Thus, the set of above equalities and inequalities determines the equilibrium correspondence for the no refund game. That is, for any (c,w,M), (1.1)-(1.3) together determine the set of triples $\{(k^*,j^*,q^*)\}$ which are Nash equilibria to the no refund game. We now proceed to prove a few simple facts about this equilibrium correspondence.

Definition: We say (k,j,w,M) is admissible if

$$0 \leq j \leq M - w$$
$$0 \leq k \leq w - 1$$

The reason for this definition is straightforward. If $j > M - w$, then it is impossible to have enough contributors to produce the good, so in equilibrium clearly $|G^3| = 0$. If $k \geq w$, then the good is always produced, so, again $|G^3| = 0$. The last condition is obvious.

Proposition 2: If (k,j,w,M) is admissible then, for any value of $c \in (0,1)$ there are at most 2 values of q such that (k,j,q) is an equilibrium. If either $k > 0$ or $j > 0$, then there is at most one such value of q.

Proof: Differentiating (1.3) with respect to q gives

$$c'(q) = \binom{M-k-j-1}{w-k-1}[(w-k-1)(1-q) - (M-w-j)q][q^{w-k-2}(1-q)^{M-w-j-1}]$$

It follows that $c(q)$ is single peaked and

$$c'(q) \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{if} \quad q \begin{matrix} < \\ = \\ > \end{matrix} \frac{w-k-1}{M-j-k-1}$$

The first part of the proposition follows immediately. To prove the second part of the proposition, first assume $k > 0$. From (1.1) and (1.3) we have

$$q \geq \frac{w-k}{M-k-j} \tag{2}$$

Since $\frac{w-k}{M-k-j} > \frac{w-k-1}{M-j-k-1}$, we have

$$1 > q > \frac{w-k-1}{M-j-k-1}.$$

However, in this region $c(q)$ is monotonic, so there is at most one value of q satisfying equation (1.3) and inequality (2). Next, suppose $j > 0$. In this case, we have, from (1.2) and (1.3)

$$q \geq \frac{w-k-1}{M-j-k-1} \tag{3}$$

A similar argument to the one above demonstrates that again there is at most one q satisfying (1.3) and inequality (3). If both $k > 0$ and $j > 0$, then only (1.3) and (2) are binding since (2) implies (3).

Corollary 2.1: If (k,j,w,M) are admissible, there exists a (k,j,q) equilibrium if and only if $c \leq c_{max}$, where

$$c_{max} = \binom{M-j-1}{w-1} \frac{(w-1)^{w-1}(M-w-j)^{M-w-j}}{(M-j-1)^{M-j-1}} \quad \text{if } k = 0 \tag{4}$$

$$= \binom{M-k-j-1}{w-k} \frac{(w-k)^{w-k}(M-w-j-1)^{M-w-j-1}}{(M-k-j-1)^{M-k-j-1}} \quad \text{if } k > 0 \tag{4'}$$

Proof: From the single-peakedness of $c(q)$ established in proposition 2, (4) is the largest value of c for which (1.3) has a solution in $q \in (0,1)$. If $k > 0$, then (1.3) and the additional restriction (2) together imply (4').

This corollary is of particular interest since it specifies a closed form solution for the maximum cost which admits a (k,j,q) equilibrium.

Corollary 2.2: For fixed (k,j,w,M) admissible with $k > 0$ or $j > 0$ and $c \leq c_{max}$, $Q'(c) < 0$, where we define $Q(c) = c^{-1}(q)$.

Proof: Immediate from proposition 2.

This corollary states that the equilibrium probability of contributing for members of G^3 is a decreasing function of cost.

In the no refund game with "small" M and fixed w, multiple equilibria abound. There are different equilibria corresponding to variation in the parameters k and j. And, for k,j fixed, there typically are multiple equilibria corresponding to permutation of membership in the three groups G^1, G^2, and G^3. The multiple equilibrium problem is less troubling for very large groups.

From Corollary 2.1, note that

$c_{max} \leq \bar{c}_{max} = \binom{M-k-j-1}{w-k-1} \frac{(w-k-1)^{w-k-1}(M-w-k-j)^{M-w-k-j}}{(M-k-j-1)^{M-k-j-1}}$. In turn, $\bar{c}_{max}$ is just the binomial probability of obtaining exactly w-k-1 successes in M-j-k-1 trials when the success probability, p, is $\frac{w-k-1}{M-j-k-1}$. It is a well-known result that the probability of obtaining any exact number of successes goes to zero as the number of trials becomes large as long as p is bounded strictly[3] away from 0 and 1. Hence, if we take a limit of $\bar{c}_{max}(M,k,j,w)$ in such a way that $p = \frac{w-k-1}{M-j-k-1}$ is bounded between 0 and 1, then the limit will equal 0. One way to do this, for example is to fix any admissible $(w^o,M^o,j^o,k^o,)$. Obtain a sequence $\{(w_n,M_n,j_n,k_n)\}_{n=1}^{\infty}$ by letting $w_n = nw^o$, $M_n = nM^o$, $j_n = nj^o$, $k_n = nk^o$. This sequence generates another sequence $\{\bar{c}^n_{max}\}_{n=0}^{\infty}$. It is easily shown that $\lim_{n\to\infty} \bar{c}^n_{max} = 0$. In other words, if we eliminate equilibria in this way, the only equilibria which are supported by positive contribution costs for all n are the pure strategy equilibria. These either have no contributors or just enough contributors to produce the good. If the public good is produced at all, it is produced with no "surplus" of contributors. By the above limiting argument, all (inefficient) mixed strategy equilibria seem to disappear in large populations.[4]

We conclude this section with an extended example in which we derive the entire set of Nash equilibria.

A Numerical Example.

Let $M = 4$, $w = 2$, $c = .096$.

Pure Strategy Equilibria: There are $\binom{4}{2} = 6$ equilibria with $k = j = 2$ and one equilibrium where no one contributes.

Equilibria With Mixed Strategies: For $k = j = 0$, we have

$$.096 = \binom{3}{1} q(1-q)^2$$

There are two solutions, $q = .8$ and $q \simeq .0343$.

Note that in one of these solutions, the expected number of contributors is 3.2, substantially greater than the efficient number of 2.0. In the other, expected contributors are $\simeq .4$, less than the efficient number. Thus, if mixed strategies were used, we might expect to see both underprovision and overprovision of the public service. Such results were reported by Simmons et al. (1981) and van de Kragt et al. (1982).

If $j = 0$, for this example, the only admissible value for $k > 0$ is $k = 1$. Thus,

$$.096 = \binom{2}{0} q^0 (1-q)^2$$

The only solution such that $0 < q < 1$ is $q \simeq .690$. Since $.690 > 1/3$, we have verified that (2) is satisfied. The expected number of contributors is 3.07.

Similarly, with $k = 0$ the only admissible value for $j > 0$ is $j = 1$. Thus,

$$.096 = \binom{2}{1} q(1-q).$$

The two solutions are $q = .949$ and
$q = .051$.

However, only $q = .949 > 1/2$ satisfies (3). Expected contributors are 2.85.

The only remaining case is $k = j = 1$. Thus,

$$.096 = \binom{2}{0} q^0 (1-q) \text{ or } q = .904.$$

Since $.904 > 1/2$, (2) is satisfied.

We find that the expected number of contributors in equilibrium can vary from 0 to 3.2. Thus, experimental findings that there are more than the efficient number of contributors cannot be taken as evidence that players are not self-interested Nash players. Indeed, equilibria with $j = 0$ are consistent with occasionally observing every player contributing!

Case II. Refund (R).

IIa. $|G^3| = 0$.

Proposition 3. The pure strategy Nash equilibria include all the equilibria specified in Proposition 1 and, if $w > 2$, equilibria where

k, $0 < k \leq w - 2$, individuals contribute and the others do not.

<u>Proof</u>: Obvious. The additional equilibria result simply because a player is indifferent between contributing and not contributing when the player is not pivotal and the public good is not provided.

<u>Remark</u>. The pure strategy Nash equilibria of R are a superset of those of $\overline{R}$. Of course, under R only the efficient (m=w) equilibria are <u>strong</u> in the sense that, if any player deviates from equilibrium, that player is strictly worse off.

We believe that this is what is meant by the observation of Simmons <u>et al</u>. that a refund removes the fear incentive for free riding. The inefficient pure strategy equilibria of the R game are <u>weak</u> in the sense that any player may unilaterally change his or her strategy without affecting anyone's payoff. In contrast, both types (m=w and m=0) are strong under $\overline{R}$. Nonetheless, even when only strong equilibria are predicted, the Nash predictions contain m=w for both types. Consequently, one would not be astonished that there is little difference in experimental results.

<u>IIb</u>. $|G^3| \neq 0$.

The equilibrium conditions analogous to (1.1)-(1.3) are:

$$c \leq \frac{\mathrm{Prob}(m^3 = w-k)}{\mathrm{Prob}(m^3 \geq w-k)} = \frac{\binom{M-k-j}{w-k} q^{w-k} (1-q)^{M-w-j}}{\sum_{t=w-k}^{M-k-j} \binom{M-k-j}{t} q^t (1-q)^{M-k-j-t}} \tag{5.1}$$

$$c \geq \frac{\mathrm{Prob}(m^3 = w-k-1)}{\mathrm{Prob}(m^3 \geq w-k-1)} = \frac{\binom{M-k-j}{w-k-1} q^{w-k-1} (1-q)^{M-w-j+1}}{\sum_{t=w-k-1}^{M-k-j} \binom{M-k-j}{t} q^t (1-q)^{M-k-j-t}} \tag{5.2}$$

$$c = \frac{\mathrm{Prob}(m_i^3 = w-k-1)}{\mathrm{Prob}(m_i^3 \geq w-k-1)} = \frac{\binom{M-k-j-1}{w-k-1} q^{w-k-1} (1-q)^{M-w-j}}{\sum_{t=w-k-1}^{M-k-j-1} \binom{M-k-j-1}{t} q^t (1-q)^{M-k-j-1-t}} \tag{5.3}$$

Note that, in each expression, the numerator is the probability of being <u>pivotal</u>, while the denominator is the probability of <u>winning</u> (the good is produced) if the player contributes. In the no refund case, in contrast, the denominator is effectively one or the probability of <u>winning or losing</u>. The refund has the effect of discarding the losing events.

<u>Proposition 4</u>. If, for given c, (5.3) has a solution, the constraints (5.1) and (5.2) are never binding.

<u>Proof</u>: Algebraic manipulations leads us to rewrite (5.1)-(5.3) as

$$c \leq 1 + \sum_{t=1}^{M-w-j} \prod_{r=1}^{t} \left(\frac{M-w-j+1-r}{w-k+r}\right) \left(\frac{q}{1-q}\right)^t \; {}^{-1} \tag{5.1'}$$

$$c \geq 1 + \sum_{t=1}^{M-w-j} \prod_{r=1}^{t} \left(\frac{M-w-j+2-r}{w-k+r-1}\right) \left(\frac{q}{1-q}\right)^t$$

$$+ 1/\binom{M-k-j}{w-k-1} \left(\frac{q}{1-q}\right)^{M-w-j+1} \; {}^{-1} \tag{5.2'}$$

$$c = 1 + \sum_{t=1}^{M-w-j} \prod_{r=1}^{t} \left(\frac{M-w-j+1-r}{w-k+r-1}\right) \left(\frac{q}{1-q}\right)^t \; {}^{-1} \tag{5.3'}$$

Since $w-k+r > w-k+r-1$ and $M-w-j+1-r < M-w-j+2-r$, we have that every term in the summation in (5.2′) is greater than every term in (5.3′) which is greater than every term in (5.1′). The extra term in (5.2′) is positive. The proposition now follows directly.

The following definition rules out the case of $j = M - w$. We return to this later.

<u>Definition</u>: (k,j,w,M) is <u>admissible for R</u> if it is admissible and $j < M - w$.

Since equation (5.3′) characterizes the entire set of equilibria for R when some players use mixed strategies, we proceed by analyzing the properties of that equation. The first observation, stated below in proposition 5 and its corollary, is that if (k,j,w,M) are admissible for R, then a unique (k,j,q) equilibrium exists for each value of c between 0 and 1. In contrast, for $\bar{R}$, equilibria existed only for $c \leq c_{max}$. This result, together with propositions 2 and 3, establishes that there are "more" equilibria under R than under $\bar{R}$.

<u>Proposition 5</u>: In (5.3′), c is a continuously differentiable function of q on (0,1). Furthermore, in this interval:

(a) $c'(q) < 0$

(b) $c(0) = 1$

(c) $\lim_{q \to 1} c(q) = 0$

<u>Proof</u>: These properties of $c(q)$ are easily obtained from (5.3′).

<u>Corollary 5.1</u>: The function $c(q)$ is invertible on (0,1) and the inverse function $Q(c)$ specifies the unique equilibrium value of q at c for given admissible k,j,w. Furthermore:

(a) $Q'(c) < 0$

(b) $Q(1) = 0$

(c) $\lim_{c \to 0} Q(c) = 1$

The next proposition states that there is greater contribution under R than under $\bar{R}$, at least in the corresponding mixed strategy equilibria. A somewhat different statement of the proposition is that for every mixed $\bar{R}$ equilibrium, there exists a corresponding (in the sense that there are the same number of pure strategy contributors and noncontributors) mixed R equilibrium with a greater expected number of contributors.

<u>Proposition 6</u>: For fixed admissible k,j,w,M, and $0 < c \leq c_{max}(k,j,w,M)$, $Q(c)$ is strictly greater than the equilibrium mixing probability under $\bar{R}$. (If $k = j = 0$, $Q(c)$ is strictly greater than both equilibrium probabilities under $\bar{R}$.)

<u>Proof</u>: Follows directly from 1.3, 5.3, and corollary 5.1.

Under $\bar{R}$ we established that no mixed strategy equilibria could be supported for large values of M,w,j,k. This is no longer the case under R, as proposition 5 states. However, there is a disappearance of these equilibria in a slightly different sense. The following proposition establishes for R that when M,w,j,k grow proportionately,[5]

$Q(c)$ converges to 0 pointwise on (0,1). That is each mixer acts essentially just like a noncontributor.

<u>Proposition 7</u>: Fix M^0. Choose admissible w^0, j^0, k^0. Denote by $Q(c;n)$ the equilibrium probability of voting function for parameters (nM^0, nw^0, nj^0, nk^0). Then for $c \in (0,1)$

$$\lim_{n \to \infty} Q(c;n) = 0$$

<u>Proof</u>: In the proof, we establish that the inverse function converges to zero. By monotonicity of the inverse function, this establishes that $Q(c;n)$ converges pointwise to 0. The numerator in (5.3) is the binomial probability of obtaining exactly $w-k-1$ successes in $M-k-j-1$ trials with fixed probability q. As $n \to \infty$, this probability goes to zero. On the other hand, the denominator is $1-F(nw-nk-nj-2;\ nM,q)$. This "tail" approaches a finite limit (given by the Normal approximation) as $n \to \infty$. Hence, $c \to 0$.

<u>The Numerical Example Continued</u> ($M = 4$, $w = 2$)

Since $w = 2$, there are few differences between R and $\bar{R}$ for this example. The pure strategy R equilibria are identical to those for $\bar{R}$ as are the mixed strategy equilibia for k=1.

When k=0 and j=0, q = .802, very close to the $\bar{R}$ value of .800. For k=0 and j=1, q=.9496, again very close to the $\bar{R}$ value of .9494. Finally, when k=0 and j=2, there are no mixed strategy equilibria.

<u>Discussion</u>. For this example, the R and $\bar{R}$ solutions are remarkably similar in both pure and mixed strategies. The only substantial differences occur for k=0, when there are no "committed" contributors. Even here, the low q solutions for $\bar{R}$ have q so low that enormous amounts of experimental data would be needed to distinguish these equilibria from the pure strategy equilibrium with no one contributing. Similarly, it would be difficult to discriminate between R and $\bar{R}$ as to their high q mixed strategy predictions. In this particular game, and perhaps more generally, it will be difficult to determine, on the basis of a small number of experiments, whether the refund rule affects the supply of public goods.

<u>Another Example</u>

We now analyze the experimental games of Simmons et al. (1981). They had M=7 and w=3 or 5. Reparameterizing their payoffs in our 0-1 metric, c=1/2. Tables 1 and 2 show the mixed strategy equilbria. Mixed strategy equilibria proliferate here for the refund condition. Consequently, even if subjects were "at equilibrium" there could be considerable scatter in the data. Moreover, mixed strategy equilibria are not entirely ruled out with no refund, even with this relatively high cost. The $\bar{R}$ probabilities are again close to certain R probabilities. A probability of 1/2 occurs in both conditions for both w values. For w=3, the k=0, j=4, $\bar{R}$ probability is .71, while the k=0, j=3 probability for R is .75. For w=5, the k=0, j=2 $\bar{R}$ probability is .84, while the k=0, j=1 probability for R is .83. As in our earlier examples, it is not easy to distinguish between the R

Table 1.

Analysis of Mixed Strategy Equilibria, M=7, w=3, c=1/2

$\bar{R}$

k \ j	0	1	2	3	4
0	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$	q = .71
1	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$
2	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$	q = .50	Not Admissible

R

k \ j	0	1	2	3	4
0	q = .34	q = .42	q = .54	q = .75	Not Admissible
1	q = .26	q = .33	q = .44	q = .67	Not Admissible
2	q = .16	q = .21	q = .29	q = .50	Not Admissible

Table 2.

Analysis of Mixed Strategy Equilibria, M=7, w=5, c=1/2

$\bar{R}$

k \ j	0	1	2
0	$c > c_{max}$	$c > c_{max}$	q = .84
1	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$
2	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$
3	$c > c_{max}$	$c > c_{max}$	$c > c_{max}$
4	$c > c_{max}$	q = .50	Not Admissible

R

k \ j	0	1	2
0	q = .66	q = .83	Not Admissible
1	q = .60	q = .80	Not Admissible
2	q = .54	q = .75	Not Admissible
3	q = .44	q = .67	Not Admissible
4	q = .29	q = .50	Not Admissible

and $\bar{R}$ conditions simply by observing the number of contributors across trials.

Inspection of Tables 1 and 2 suggests:

Proposition 8. The equilibrium probability q is increasing in j and decreasing in k in R games.

Proof:

1. Increasing k for fixed j makes every term in the product in (5.3') larger. Thus, for fixed c, $\frac{q}{1-q}$ and hence q must decrease.
2. Increasing j for fixed k drops one positive term from the sum in (5.3') and makes every term in the product smaller. Hence, for fixed c, q must increase.

Proposition 9. If $w < M$, $M \geq 2$, there exists at least one mixed strategy equilibrium to every R game for all $c \in (0,1)$ and to every $\bar{R}$ game for all $c \in (0,1/2)$.

Proof:

1. Consider $j = M-w-1$, $k = w-1$.
2. Then from (1.3) or (5.3), $q = 1-c$.
3. The constraint (2) is satisfied if $q \geq 1/2$ or $c \leq 1/2$.

Unanimity Rule. A unanimity rule, where $w = M$, is one extreme case which is of particular interest because it sharply illustrates this difference between the R and $\bar{R}$ rules. As others have pointed

out, rebates remove the "fear" motivation to free ride. Unanimity removes the greed motivation as well, since if $w = M$, there is no possibility of "overcontribution." This leads to a situation in which there is no strong motivation under the refund rule for a member of the group not to contribute. In fact, consider the case $j = M-w$. Then, it is easy to show that, with refund, the only equilibria are those with $k = M-j$ and, hence, $|G^3| = 0$. There are no mixers. Indeed, if $M = w$, so there is a unanimity rule, unanimous contribution constitutes a dominant strategy equilibrium (i.e. regardless what other members do, each member has a best response of contributing). With no refund the "fear" incentive remains, and there are exactly three Nash equilibria, none of which is a dominant strategy equilibrium. These equilibria are stated below.

Proposition 10. If $w=M > 2$, there are only three equilibria in the $\bar{R}$ game: 1. Pure strategy equilibrium with m=0. 2. Pure strategy equilibrium with m=M. 3. Mixed strategy equilibrium with j=k=0 and $q = c^{\frac{1}{M-1}}$.

Proof:

1. The pure strategy equilibria are established by proposition 1.
2. There are obviously no mixed strategy equilibria with $j \neq 0$.
3. If $j = 0$, and $k = 0$, $q = c^{\frac{1}{M-1}}$ follows from (1.3).
4. If $k \neq 0$, then (2) has $q \geq (M-k)/(M-k) = 1$, so no other mixed

strategy equilibrium exists.

III. SURVEY OF EXPERIMENTAL EVIDENCE

Much of the experimental literature dealing with discrete public goods makes no attempt to control for or specify the costs facing the participants. Consequently, the results are rarely germane to a test of our theory. A large set of these experiments were aimed at testing the qualitative predictions of Olson (1968) that collective goods were less likely to be provided and that individual contribution rates would fall as group size increased.

For example, Darley and Latané (1968) conducted social-psychological experiments in which subjects were led to believe that an individual had had a nervous seizure and that they were in a group of size M, with M being set to 1, 2, or 5. The experimenters were interested in whether a subject reported the seizure or, in our terms, a w=1 game. Indeed a lower proportion of subjects reported the seizure as group size increased. Sweeney (1974) offerred the completely ad hoc suggestion that the probability of reporting (contributions) would be given as $q = y/M^{-z}$, where y and z are arbitrary positive constants. This "law" would agree with our result for the pure strategy equilibrium where the frequency of contributors is 1/M, but it is quite distinct from the mixed strategy result that $q = 1 - c^{\frac{1}{M-1}}$. Table 3 shows that the results are reasonably consistent with a game-theoretic approach although the lack of 100 percent contribution when M=1 would suggest, at the least, that not

all participants perceived the same cost.

Variations in cost do need to be considered by analysts. Sweeney goes on to argue that the Darley and Latané results, which show that the probability at least one individual will contribute is about .85, independent of M, is inconsistent with real-world observations such as the 1964 Genovese murder in New York. In that case, none of 38 individuals contributed (called the police). But for these individuals one can not rule out that c>1 which would make not contributing a dominant strategy. (Calling the police may be costless but appearing as a witness in a lengthy assualt or murder trial is very costly.)

Table 3. The Darley-Latané Experiments

Group-Size M	% Contributors Actual Data	% Expected Contributors Pure Strategy Equilibrium	% Expected, Contributors Mixed Strategy Equilibrium with c=.3	Number of Subject
1	85	100	-	13
2	62	50	70	26
5	31	20	26	13

Source: Adapted from Sweeney (1974), p. 268.

Experiments that, since costs are controlled, bear directly on the preceeding theory are found in 34 non-repeated M=7 games reported by Simmons et al. (1981) and van de Kragt et al. (1982). These experiments consisted of 7 with w=3 and refund, 10 with w=3 and no refund, 7 with w=5 and refund, and 10 with w=5 and no refund. Normalizing their payoffs shows c=1/2.

Consider the 8x4 contingency table (Table 4) that cross-classifies the experiments by number contributing. Treating the experimental condition as the independent variable and denoting the number contributing by a, we can list the following possible a priori predictions from our game-theoretic analysis.

$\mathbb{P}_1$: All pure strategy Nash equilibria are predicted.

If w = 3 and R, a = 0, 1, or 3

w = 3 and $\overline{R}$, a = 0 or 3

w = 5 and R, a = 0, 1, 2, 3, or 5

w = 5 and $\overline{R}$, a = 0 or 5

$\mathbb{P}_2$: All pure strategy Nash equilibrium that are not weak are predicted.

If w = 3 and R, a = 3

w = 5 and $\overline{R}$, a = 0 or 3

w = 5 and R, a = 5

w = 5 and $\overline{R}$, a = 0 or 5

$\mathbb{P}_3$: Any outcome consistent with either a mixed or pure Nash equilibrium is predicted.[6]

If w = 3 and R, a = 0, 1, 2, 3, 4, 5, 6, or 7

If w = 3 and $\overline{R}$, a = 0, 1, 2, 3, or 4

If w = 5 and R, a = 0, 1, 2, 3, 4, 5, 6, or 7

If w = 5 and $\overline{R}$, a = 1, 2, 3, 4, 5, or 6

To evaluate these predictions, we use the ∇ ($-\infty < \nabla \leq 1$) measure of predictions success (Hildebrand et al., 1977).[7] We also use the U ($0 \leq U \leq 1$) measure of prediction precision. (Since $\mathbb{P}_2$ must make at least as many errors as $\mathbb{P}_1$, $\mathbb{P}_2$ is more precise than $\mathbb{P}_1$. Similarly, $\mathbb{P}_1$ is more precise than $\mathbb{P}_3$.) The results appear in Table 5. All ∇'s are positive, but we would reject the null hypothesis that $\nabla = 0$ using an .05 level of significance only for $\mathbb{P}_2$. However, the ∇ value of .09 for $\mathbb{P}_2$ is quite low. The value for $\mathbb{P}_3$, .32, is more substantial, but the precision of the prediction is too low to permit the application of the asymptotic sampling theory developed by Hildebrand et al. (1977).

Simmons et al. (1981) believed there would be more contributions with a refund because the "fear" motivation for free-riding has been eliminated. Were these authors to have combined their social-psychological concerns with game theory, they might have arrived at the proposition that under no refund either the efficient Nash equilibrium number of contributors or fewer result while with a refund the efficient number or more contribute. This leads to:

$\mathbb{P}_4$: Efficiency with "fear" considerations

If w = 3 and R, a = 3, 4, 5, 6, or 7

w = 3 and $\overline{R}$, a = 0, 1, 2, or 3

w = 5 and R, a = 5, 6, or 7

w = 5 and $\overline{R}$, a = 1, 2, 3, 4, or 5

This proposition, as an a priori prediction, would be reasonably sucessful, since ∇ = .32 and we strongly reject the null hypothesis of ∇ = 0.[8] Despite this success, precision is modest (U = .399). None of the four predictions represents a strong explanation of the experimental outcomes.

Table 4.

The Simmons Experiments

Number of Contributors	Conditions: w=3, R	w=3, $\bar{R}$	w=5, R	w=5, $\bar{R}$
0	0	0	0	0
1	0	1	0	0
2	0	2	0	0
3	2	3	1	1
4	2	1	2	5
5	2	1	3	2
6	1	1	1	2
7	0	1	0	0
Total	7	10	7	10 N=34

Table 5.

Prediction Analysis

	$\hat{\nabla}$	Estimated std.error	Estimated $\hat{U}$
$\mathbb{P}_1$	.051	.035	.713
$\mathbb{P}_2$	.094	.039	.779
$\mathbb{P}_3$	.320	.076	.130
$\mathbb{P}_4$	.336	.002	.399

Note: $\mathbb{P}_4$ is significant at .01 level, $\mathbb{P}_2$ at .05. In $\mathbb{P}_3$, total errors are too few to apply the asymptotic tests.

What may underlie the variations in Simmons' experiments is the multiplicity of Nash equilibria. For example, in the w = 3 game, there are 35 efficient pure strategy equilibrium, and every player is a contributor in 15 such equilibria. Given the one shot character of the games, participants have few guides to action.

IV. CONCLUSION

This paper analyzed two different voluntary contribution schemes for the provision of a discrete public good by modelling these schemes as two different rules for a one-team participation game. The Nash equilibria in the two games were compared and contrasted, for games played by both small groups and very large groups. The results for small groups were then applied to an analysis of experimental data on free riding behavior. The _a priori_ predictions of the game theoretic model were only moderately successful in explaining the experimental outcomes. We believe there are at least two reasons for this. First, because of the presence of multiple equilibria, the range of possible observed behavior consistent with equilibrium behavior was very wide. Second, since the games were played only once, subjects had no chance to learn about the behavior of other players. Because of the multiplicity of equilibria, learning or some form of coordination is probably very important (possibly necessary) for the attainment of a Nash equilibrium. In light of these extreme coordination problems, we find it remarkable that the game theoretic predictions were as successful as they turned out to be.

While we did not apply the results for large groups to experimental or field data, a number of interesting theoretical results emerged. First, for large groups, the two rules differed only in the predicted _pure strategy_ equilibria, because, under both rules, mixed strategy equilibria "disappear." This disappearance takes two different forms. Either the mixing probabilities converge to 0 or 1 ("essentially" pure strategies) or mixed strategy equilibria are not supported for positive costs.

For both large and small groups, it was found that equilibria under the refund rule will lead to greater expected contribution than equilibria under the no refund rule. However, this difference does not seem to be as large as previous researchers have suggested. This rather surprising result has been borne out by experimental data.

The results of this paper for the no refund rule can be readily combined with those for two team games reported in Palfrey and Rosenthal (1982). There, given majority rule, w was, essentially, zero. Expected participation with large number of players equalled either zero or w + twice the size of the minority. In the one team games with no refund, the minority has size zero, and thus expected participation is also either zero or w + twice the size of the minority. The result of "zero or w + twice the minority" should generalize to two team games with supramajority voting.

FOOTNOTES

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1. In Palfrey and Rosenthal (1982), one variant gave a payoff of 1/2 to everyone even if no one volunteered to provide the good. In the one team case, this variant does not apply.

2. Chamberlin (1974) and McGuire (1974) have examined a rule similar to the no refund rule for the provision of <u>continuous</u> rather than discrete public goods.

3. By bounded strictly away from 0 and 1, we mean that there exist constants A and B such that $0 < A < p < B < 1$.

4. Not quite all mixed strategy equilibria disappear in large populations. A few special cases persist. To see this, suppose we take limits as above except we let $k_n = 0$ for all n and have $j_n = m_n - w_n - 1$. Then c_{max} approaches $\frac{1}{e}$ in the limit, and for all $c \varepsilon (0, \frac{1}{e})$, Q(c) converges to 1. Thus with large populations, for relatively low cost, these equilibria survive, but mixers behave almost as if they are using a pure strategy of voting. It is tedious, but straightforward to prove more generally that in all those mixed strategies equilibria which can be supported by positive costs in large populations, Q(c) converge to either 0 or

1. Hence, in a different sense, these mixed strategies "disappear" as well.

5. If limits are taken "nonproportionately," we find a few exceptions similar to the example given in n. 4.

6. From Tables 1 and 2, we note that the presence of an equilibrium with j=k=0 implies that all outcomes can occur under R. Under $\overline{R}$, the predictions result from noting that at least 3 people must abstain with w=3 and at least one must abstain with w=5.

7. A general proportionate-reduction-in-error measure, ∇ is analogous to R^2. In fact, R^2 is a special case of ∇ (see Hildebrand et al., 1981.) Because the experiments controlled the independent variable, we have used the variance expression found in Hildebrand et al., 1977, p. 202.

8. It is interesting that Simmons et al. concluded that the value of w and the refund conditions had no influence on contributions. These authors used a standard chi-square test which is equivalent to testing the <u>a priori</u> theory "the data are not statistically independent." This test fails to exploit the directionality of <u>a priori</u> theory like $\mathbb{P}_1 - \mathbb{P}_4$. The chi-square test is thus very unlikely to reject the null hypothesis.

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